

# SUMS OF ZEROS FOR CERTAIN SPECIAL FUNCTIONS

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**ABSTRACT.** In this work we evaluate sums of the zeros for the Bessel function  $J_\nu(z)$ , the Airy function  $A(z)$ , the Riemann zeta function  $\zeta(z)$ ,  $L$ -series  $L(s, \chi)$  with real primitive characters, Ramanujan's entire function (a.k.a.  $q$ -Airy function)  $A_q(z)$ ,  $q$ -Bessel function  $J_\nu^{(2)}(z; q)$ .

## 1. INTRODUCTION

Given an entire function  $f(z)$ , it is interesting to find formulas for various sums of zeros for  $f(z)$ . Generally speaking, it is relatively easy to calculate the multiple sums and hard to find the related power sums. In this work we prove an identity for certain class of entire functions that is very similar to the formulas for polynomials. This class includes the Bessel function  $J_\nu(z)$ , the Riemann zeta function  $\zeta(z)$ ,  $L$ -series  $L(s, \chi)$  with real primitive characters, Ramanujan's entire function (a.k.a.  $q$ -Airy function)  $A_q(z)$ ,  $q$ -Bessel function  $J_\nu^{(2)}(z; q)$ . Using this identity we give closed form evaluations of certain multiple sums and power sums of the zeros of these functions.

The work is divided into four sections, in section 2 we present some facts on the special functions; We state and derive the identity in section 3; In the section 4 we apply the identity to some special functions.

## 2. PRELIMINARIES

**2.1. Bessel type functions** [1, 3, 7, 13, 15]. For any complex number  $z \in \mathbb{C}$ , the Euler's  $\Gamma(z)$  is defined as

$$(2.1) \quad \frac{1}{\Gamma(z)} := z \prod_{j=1}^{\infty} \left(1 + \frac{z}{j}\right) \left(1 + \frac{1}{j}\right)^{-z}.$$

The Bessel functions  $J_\nu(z)$  is defined as

$$(2.2) \quad J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n + \nu + 1)n!} \left(\frac{z}{2}\right)^{\nu+2n}.$$

It is well known that, for  $\nu > -1$ , the zeros of the even entire function  $z^{-\nu}J_\nu(z)$  are real and simple. We denote the positive zeros as

$$(2.3) \quad 0 < j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n}, \dots$$

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They satisfy the following identity

$$(2.4) \quad \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu+1)}.$$

The Bessel functions have the infinite product expansion

$$(2.5) \quad J_{\nu}(z) = \frac{z^{\nu}}{2^{\nu}\Gamma(\nu+1)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,n}^2}\right).$$

The Bessel function  $J_{\nu}(z)$  also satisfies the following second order differential equation

$$(2.6) \quad z^2 \frac{d^2 y(z)}{dz^2} + z \frac{dy(z)}{dz} + (z^2 - \nu^2)y(z) = 0.$$

The Airy function is defined as [13]

$$(2.7) \quad A(z) := \frac{\pi}{3} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+2/3)n!} \left(-\frac{z}{3}\right)^{3n} + \frac{\pi z}{9} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+4/3)n!} \left(-\frac{z}{3}\right)^{3n}.$$

Clearly [1, 12],

$$(2.8) \quad A(z) = \frac{\pi}{\sqrt[3]{3}} \text{Ai} \left( -\frac{z}{\sqrt[3]{3}} \right).$$

The Airy function  $A(z)$  satisfies the following second order differential equation

$$(2.9) \quad y''(z) + \frac{z}{3}y(z) = 0.$$

It is known that  $A(z)$  has infinitely many real zeros, all of them are positive and simple. Let us denote them as

$$(2.10) \quad 0 < i_1 < i_2 < \dots$$

It is also known that

$$(2.11) \quad i_{\nu} \sim \nu^{2/3}$$

as  $\nu \rightarrow \infty$ .

From the infinite product expansion for  $\text{Ai}(z)$  [12] we get

$$(2.12) \quad A(z) = \frac{\pi}{3} \frac{e^{\kappa z}}{\Gamma(2/3)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{i_n}\right) e^{z/i_n},$$

where

$$(2.13) \quad \kappa^2 := \sum_{n=1}^{\infty} \frac{1}{i_n^2} = \frac{3\Gamma(2/3)^4}{4\pi^2}, \quad k > 0.$$

Thus,

$$(2.14) \quad \frac{9\Gamma(2/3)^2 A(\sqrt{z})A(-\sqrt{z})}{\pi^2} = \prod_{n=1}^{\infty} \left(1 - \frac{z}{i_n^2}\right).$$

From the product formula

$$(2.15) \quad \text{Ai}(z)\text{Ai}(-z) = \frac{1}{\sqrt[3]{2\pi}} \int_{-\infty}^{\infty} \text{Ai}(\sqrt[3]{4t^2}) e^{2izt} dt$$

we obtain

$$(2.16) \quad \prod_{n=1}^{\infty} \left(1 - \frac{z}{i_n^2}\right) = 2^{5/3} 3^{4/3} \Gamma\left(\frac{2}{3}\right)^2 \int_0^{\infty} \text{Ai}(2^{2/3} t^2) \cos\left(\frac{2t z^{1/2}}{\sqrt[3]{3}}\right) dt$$

$$= 2^{5/3} 3^{4/3} \Gamma\left(\frac{2}{3}\right)^2 \sum_{n=0}^{\infty} \frac{(-1)^n m_n}{(2n)!} \left(\frac{2}{\sqrt[3]{3}}\right)^{2n} z^n,$$

where

$$(2.17) \quad m_n := \int_0^{\infty} \text{Ai}(2^{2/3} t^2) t^{2n} dt, \quad n \in \mathbb{N} \cup \{0\}.$$

From

$$(2.18) \quad \text{Ai}\left(\left(\frac{3x}{2}\right)^{2/3}\right) = \frac{1}{\pi\sqrt[3]{3}} \left(\frac{3x}{2}\right)^{2/3} K_{1/3}(x)$$

we obtain

$$(2.19) \quad \text{Ai}(z) = \frac{z^{1/2}}{\pi\sqrt[3]{3}} K_{1/3}\left(\frac{2z^{3/2}}{3}\right).$$

Hence,

$$(2.20) \quad m_n = \frac{3^{(2n-1)/3-1/2}}{\pi 2^{(4n)/3+1}} \int_0^{\infty} K_{1/3}(u) u^{(2n-1)/3} du$$

$$= \frac{3^{(2n-1)/3-1/2}}{\pi 2^{(2n+7)/3}} \Gamma\left(\frac{n}{3} + \frac{1}{6}\right) \Gamma\left(\frac{n}{3} + \frac{1}{2}\right).$$

Then,

$$(2.21) \quad \prod_{n=1}^{\infty} \left(1 - \frac{z}{i_n^2}\right) = \sum_{n=0}^{\infty} (-1)^n \alpha_n z^n,$$

where

$$(2.22) \quad \alpha_n := \frac{\sqrt[3]{3} \Gamma\left(\frac{2}{3}\right)^2}{\sqrt[3]{4}\pi} \frac{16^{n/3} \Gamma\left(\frac{n}{3} + \frac{1}{6}\right) \Gamma\left(\frac{n}{3} + \frac{1}{2}\right)}{(2n)!}.$$

2.2. *q*-Series [3, 8]. Assume that  $0 < q < 1$ , let

$$(2.23) \quad (z; q)_{\infty} := \prod_{n=0}^{\infty} (1 - zq^n),$$

$$(2.24) \quad (z; q)_n := \frac{(z; q)_{\infty}}{(zq^n; q)_{\infty}},$$

and

$$(2.25) \quad (z_1, z_2, \dots, z_m; q)_n := \prod_{j=1}^m (z_j; q)_n$$

for any  $m \in \mathbb{N}, n \in \mathbb{Z}$  and  $z, z_1, z_2, \dots, z_m \in \mathbb{C}$ .

The *q*-Bessel functions  $J_{\nu}^{(2)}(z; q)$  is defined as

$$(2.26) \quad J_{\nu}^{(2)}(z; q) := \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(-q^{n+\nu})^n}{(q, q^{\nu+1}; q)_n} \left(\frac{z}{2}\right)^{\nu+2n}.$$

It is known that for  $\nu > -1$ , all the zeros of  $z^{-\nu} J_\nu^{(2)}(z; q)$  are real and simple. We denote the positive zeros as

$$(2.27) \quad 0 < j_{\nu,1}(q) < j_{\nu,2}(q) < \cdots < j_{\nu,n}(q) < \dots$$

It is known that [9, 11]

$$(2.28) \quad \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}(q)} < \infty.$$

Then,

$$(2.29) \quad \left(\frac{z}{2}\right)^{-\nu} J_\nu^{(2)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,n}^2(q)}\right).$$

The Ramanujan's entire function (a.k.a.  $q$ -Airy function)  $A_q(z)$  is defined as [3, 11]

$$(2.30) \quad A_q(z) := \sum_{n=0}^{\infty} \frac{q^{n^2} (-z)^n}{(q; q)_n}.$$

It is known that all the zeros of  $A_q(z)$  are positive and simple, we let them be

$$(2.31) \quad 0 < i_1(q) < i_2(q) < \cdots < i_n(q) < \dots$$

It is also known that

$$(2.32) \quad \sum_{n=1}^{\infty} \frac{1}{i_n(q)} < \infty.$$

Hence,

$$(2.33) \quad A_q(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{i_n(q)}\right).$$

**2.3. The Riemann Zeta Function**  $\zeta(s)$  [1, 3, 6, 10, 14]. The Riemann zeta function  $\zeta(s)$  is defined as

$$(2.34) \quad \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1.$$

The entire functions

$$(2.35) \quad \xi(z) := \frac{z(z-1)}{2\pi^{z/2}} \Gamma\left(\frac{z}{2}\right) \zeta(z),$$

and

$$(2.36) \quad \Xi(z) := \xi\left(\frac{1}{2} + iz\right)$$

are of order 1. They satisfy the following functional equations,

$$(2.37) \quad \xi(z) = \xi(1-z),$$

and

$$(2.38) \quad \Xi(z) = \Xi(-z).$$

The function  $\Xi(z)$  has an integral representation

$$(2.39) \quad \Xi(z) = \int_0^\infty \phi(t) \cos(zt) dt,$$

where

$$(2.40) \quad \phi(t) := 4\pi \sum_{n=1}^{\infty} \left\{ 2\pi n^4 e^{-9t/2} - 3n^2 e^{-5t/2} \right\} \exp(-n^2 \pi e^{-2t}).$$

Evidently,

$$(2.41) \quad \Xi(z) = \sum_{n=0}^{\infty} \frac{(-1)^n b_n}{(2n)!} z^{2n},$$

where

$$(2.42) \quad b_n := \int_0^{\infty} t^{2n} \phi(t) dt.$$

It is well known that  $\phi(t)$  is positive, even and fast decreasing on  $\mathbb{R}$ . From formula (2.42), it is clear that

$$(2.43) \quad b_0 = \Xi(0) > 0,$$

we list all its zeros with positive real part first according to their real parts, then their imaginary parts,

$$(2.44) \quad z_1, z_2, \dots, z_n, \dots$$

Then,

$$(2.45) \quad \Xi(z) = b_0 \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{z_n^2} \right).$$

Thus,

$$(2.46) \quad \prod_{n=1}^{\infty} \left( 1 - \frac{z}{z_n^2} \right) = \frac{\Xi(\sqrt{z})}{b_0} = \sum_{n=0}^{\infty} (-1)^n \beta_n z^n,$$

where

$$(2.47) \quad \beta_0 := 1, \quad \beta_n := \frac{b_n}{(2n)! b_0}, \quad n \in \mathbb{N}.$$

More generally, let  $\chi(n)$  be a real primitive character to modulus  $m$ . The function  $L(s, \chi)$  is defined as

$$(2.48) \quad L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re(s) > 1.$$

Let

$$(2.49) \quad a := \begin{cases} 0, & \chi(-1) = 1 \\ 1, & \chi(-1) = -1 \end{cases},$$

it is known that for

$$(2.50) \quad \xi(s|\chi, a) := \left( \frac{\pi}{m} \right)^{-(s+a)/2} \Gamma\left( \frac{s+a}{2} \right) L(s, \chi),$$

then,

$$(2.51) \quad \xi(1-s|\chi, a) = \xi(s|\chi, a).$$

It is also known that the entire function

$$(2.52) \quad \Xi(z|\chi, a) := \xi\left( \frac{1}{2} + iz|\chi, a \right)$$

is of order 1, even and has an integral representation

$$(2.53) \quad \Xi(z|\chi, a) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-izt} \phi(t|\chi, a) dt,$$

where

$$(2.54) \quad \phi(t|\chi, 0) := 2e^{-t/2} \sum_{n=-\infty}^{\infty} \chi(n) e^{-n^2 \pi e^{-2t}/m},$$

and

$$(2.55) \quad \phi(t|\chi, 1) := 2e^{-3t/2} \sum_{n=-\infty}^{\infty} n\chi(n) e^{-n^2 \pi e^{-2t}/m}.$$

From  $a = 0$ ,

$$(2.56) \quad \sum_{n=-\infty}^{\infty} \chi(n) e^{-n^2 \pi x/m} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} \chi(n) e^{-n^2 \pi/(mx)}, \quad x > 0,$$

and  $a = 1$ ,

$$(2.57) \quad \sum_{n=-\infty}^{\infty} n\chi(n) e^{-n^2 \pi x/m} = x^{-3/2} \sum_{n=-\infty}^{\infty} n\chi(n) e^{-n^2 \pi/(mx)}, \quad x > 0.$$

It is easy to verify that

$$(2.58) \quad \phi(-t|\chi, a) = \phi(t|\chi, a), \quad t \in \mathbb{R}.$$

Then,

$$(2.59) \quad \begin{aligned} \Xi(z|\chi, a) &= \int_0^{\infty} \phi(t|\chi, a) \cos(zt) dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n b_n(\chi, a) z^{2n}}{(2n)!}, \end{aligned}$$

where

$$(2.60) \quad b_n(\chi, a) := \int_0^{\infty} t^{2n} \phi(t|\chi, a) dt.$$

Assume that

$$(2.61) \quad b_0(\chi, a) = \Xi(0|\chi, a) = \xi \left( \frac{1}{2} | \chi, a \right) \neq 0,$$

then formula (2.59) says that 0 is not a zero of  $\Xi(z|\chi, a)$ . We list all its zeros with positive real part first according to their real parts, then their imaginary parts,

$$(2.62) \quad z_1(\chi, a), z_2(\chi, a), \dots, z_n(\chi, a), \dots$$

Then,

$$(2.63) \quad \Xi(z|\chi, a) = b_0(\chi, a) \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{z_n(\chi, a)^2} \right).$$

Thus,

$$(2.64) \quad \prod_{n=1}^{\infty} \left( 1 - \frac{z}{z_n(\chi, a)^2} \right) = \frac{\Xi(\sqrt{z}|\chi, a)}{b_0(\chi, a)} = \sum_{n=0}^{\infty} (-1)^n \beta_n(\chi, a) z^n,$$

where

$$(2.65) \quad \beta_0(\chi, a) := 1, \quad \beta_n(\chi, a) := \frac{b_n(\chi, a)}{(2n)!b_0(\chi, a)}, \quad n \in \mathbb{N}.$$

### 3. MAIN RESULTS

**Theorem 3.1.** *Given a sequence of non-zero complex numbers  $\{\lambda_n\}_{n=1}^\infty \subset \mathbb{C}$  satisfying*

$$(3.1) \quad \sum_{n=1}^{\infty} |\lambda_n| < \infty.$$

Let

$$(3.2) \quad f(z) := \prod_{n=1}^{\infty} (1 - z\lambda_n) = \sum_{n=0}^{\infty} (-1)^n \sigma_n z^n,$$

then

$$(3.3) \quad \frac{(-1)^n f^{(n)}(z)}{n! f(z)} = \sum_{1 \leq k_1 < k_2 < \dots < k_n} \frac{\lambda_{k_1} \lambda_{k_2} \dots \lambda_{k_n}}{(1 - z\lambda_{k_1})(1 - z\lambda_{k_2}) \dots (1 - z\lambda_{k_n})}$$

for all  $n \in \mathbb{N}$  and all  $z \in \mathbb{C}$  which is not in the sequence  $\{\lambda_n^{-1}\}_{n=1}^\infty$ . In particular, we have

$$(3.4) \quad \sigma_0 = 1, \quad \sigma_1 = \sum_{k=1}^{\infty} \lambda_k,$$

and

$$(3.5) \quad \sigma_n = \frac{(-1)^n f^{(n)}(0)}{n!} = \sum_{1 \leq k_1 < k_2 < \dots < k_n} \lambda_{k_1} \lambda_{k_2} \dots \lambda_{k_n}.$$

Let

$$(3.6) \quad s_n := \sum_{k=1}^{\infty} \lambda_k^n, \quad n \in \mathbb{N},$$

then for  $n \in \mathbb{N}$  we have

$$(3.7) \quad s_n = (-1)^{n-1} n \sigma_n + \sum_{j=1}^{n-1} (-1)^{j-1} \sigma_j s_{n-j}$$

and

$$(3.8) \quad \frac{s_n}{c^n} = \det \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \frac{\sigma_1}{c} \\ -\frac{\sigma_1}{c} & 1 & 0 & \dots & 0 & -\frac{2\sigma_2}{c^2} \\ \frac{\sigma_2}{c^2} & -\frac{\sigma_1}{c} & 1 & \dots & 0 & \frac{3\sigma_3}{c^3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(-1)^{n-2} \sigma_{n-2}}{c^{n-2}} & \frac{(-1)^{n-3} \sigma_{n-3}}{c^{n-3}} & \frac{(-1)^{n-4} \sigma_{n-4}}{c^{n-4}} & \dots & 1 & \frac{(-1)^{n-2} (n-1) \sigma_{n-1}}{c^{n-1}} \\ \frac{(-1)^{n-1} \sigma_{n-1}}{c^{n-1}} & \frac{(-1)^{n-2} \sigma_{n-2}}{c^{n-2}} & \frac{(-1)^{n-3} \sigma_{n-3}}{c^{n-3}} & \dots & -\frac{\sigma_1}{c} & \frac{(-1)^{n-1} n \sigma_n}{c^n} \end{pmatrix},$$

for any  $c \neq 0$ .

*Proof.* Clearly, the condition (3.1) implies that (3.2) and (3.3) converge absolutely and uniformly on any compact subset of  $\mathbb{C}$ . Thus,  $f(z)$  is an entire function. Then we have

$$(3.9) \quad (-1)f'(z) = f(z) \sum_{n=1}^{\infty} \frac{\lambda_n}{1 - z\lambda_n}.$$

Assume that (3.3) is true for some positive integer  $n$ , or

$$(3.10) \quad (-1)^n f^{(n)}(z) = n!f(z) \times \sum_{1 \leq k_1 < k_2 \dots < k_n} \frac{\lambda_{k_1} \lambda_{k_2} \dots \lambda_{k_n}}{(1 - z\lambda_{k_1})(1 - z\lambda_{k_2}) \dots (1 - z\lambda_{k_n})},$$

then,

$$(3.11) \quad \begin{aligned} & (-1)^{n+1} f^{(n+1)}(z) = (-1)n!f'(z) \\ & \times \sum_{1 \leq k_1 < k_2 \dots < k_n} \frac{\lambda_{k_1} \lambda_{k_2} \dots \lambda_{k_n}}{(1 - z\lambda_{k_1})(1 - z\lambda_{k_2}) \dots (1 - z\lambda_{k_n})} \\ & - n!f(z) \sum_{1 \leq k_1 < k_2 \dots < k_n} \frac{\lambda_{k_1}^2 \lambda_{k_2} \dots \lambda_{k_n}}{(1 - z\lambda_{k_1})^2 (1 - z\lambda_{k_2}) \dots (1 - z\lambda_{k_n})} \\ & - n!f(z) \sum_{1 \leq k_1 < k_2 \dots < k_n} \frac{\lambda_{k_1} \lambda_{k_2}^2 \dots \lambda_{k_n}}{(1 - z\lambda_{k_1})(1 - z\lambda_{k_2})^2 \dots (1 - z\lambda_{k_n})} \\ & - \dots \\ & - n!f(z) \sum_{1 \leq k_1 < k_2 \dots < k_n} \frac{\lambda_{k_1} \lambda_{k_2} \dots \lambda_{k_n}^2}{(1 - z\lambda_{k_1})(1 - z\lambda_{k_2}) \dots (1 - z\lambda_{k_n})^2} \\ & = n!f(z) \left\{ \sum_{1 \leq k_1 < k_2 \dots < k_n} \sum_{k=1}^{\infty} \frac{\lambda_k}{1 - z\lambda_k} \frac{\lambda_{k_1} \lambda_{k_2} \dots \lambda_{k_n}}{(1 - z\lambda_{k_1})(1 - z\lambda_{k_2}) \dots (1 - z\lambda_{k_n})} \right. \\ & - \sum_{1 \leq k_1 < k_2 \dots < k_n} \frac{\lambda_{k_1}^2 \lambda_{k_2} \dots \lambda_{k_n}}{(1 - z\lambda_{k_1})^2 (1 - z\lambda_{k_2}) \dots (1 - z\lambda_{k_n})} \\ & - \sum_{1 \leq k_1 < k_2 \dots < k_n} \frac{\lambda_{k_1} \lambda_{k_2}^2 \dots \lambda_{k_n}}{(1 - z\lambda_{k_1})(1 - z\lambda_{k_2})^2 \dots (1 - z\lambda_{k_n})} \\ & - \dots \\ & \left. - \sum_{1 \leq k_1 < k_2 \dots < k_n} \frac{\lambda_{k_1} \lambda_{k_2} \dots \lambda_{k_n}^2}{(1 - z\lambda_{k_1})(1 - z\lambda_{k_2}) \dots (1 - z\lambda_{k_n})^2} \right\}. \end{aligned}$$

The first sum within the braces could be split into several sums according to whether  $k$  equals one of these  $k_1, \dots, k_n$  or in one of the  $n+1$  intervals:

$$(3.12) \quad (1, k_1), (k_1, k_2), \dots, (k_n, \infty).$$

It is clear that the  $n$  sums in the first case cancel out the last  $n$  negative sums within the braces, while each of the  $n+1$  sums obtained from the last case, after



renaming the dummy variables, equals

$$(3.13) \quad \sum_{1 \leq k_1 < k_2 < \dots < k_{n+1}} \frac{\lambda_{k_1} \lambda_{k_2} \dots \lambda_{k_{n+1}}}{(1 - z \lambda_{k_1})(1 - z \lambda_{k_2}) \dots (1 - z \lambda_{k_{n+1}})},$$

which implies that (3.3) holds for  $n + 1$ , and the proof of (3.2) is finished by the principle of induction. Observe that

$$(3.14) \quad -f'(z) = \sum_{n=0}^{\infty} (-1)^n (n+1) \sigma_{n+1} z^n$$

and

$$(3.15) \quad \sum_{n=1}^{\infty} \frac{\lambda_n}{1 - z \lambda_n} = \sum_{n=0}^{\infty} s_{n+1} z^n,$$

equation (3.9) becomes

$$(3.16) \quad \sum_{n=0}^{\infty} (-1)^n (n+1) \sigma_{n+1} z^n = \left( \sum_{n=0}^{\infty} (-1)^n \sigma_n z^n \right) \left( \sum_{n=0}^{\infty} s_{n+1} z^n \right).$$

We get (3.7) by equating the corresponding coefficients of  $z^n$ . (3.8) is obtained from (3.7) by solving for  $\left\{ \frac{s_1}{c}, \frac{s_2}{c^2}, \dots, \frac{s_n}{c^n} \right\}$  with Cramer's rule.  $\square$

#### 4. APPLICATIONS

4.1. **Sine function**  $\sin(z)$ . In the Theorem 3.1 we take

$$(4.1) \quad \lambda_k = \frac{1}{k^2}, \quad k \in \mathbb{N},$$

then,

$$(4.2) \quad f(z) = \prod_{k=1}^{\infty} \left( 1 - \frac{z}{k^2} \right) = \frac{\sin(\pi \sqrt{z})}{\pi \sqrt{z}}.$$

Since

$$(4.3) \quad \frac{\sin(\pi \sqrt{z})}{\pi \sqrt{z}} = \sum_{n=0}^{\infty} \frac{\pi^{2n}}{(2n+1)!} (-z)^n,$$

then,

$$(4.4) \quad \sum_{1 \leq k_1 < k_2 < \dots < k_n} \frac{1}{k_1^2 \cdot k_2^2 \dots k_n^2} = \frac{\pi^{2n}}{(2n+1)!},$$

which is known, see [5]. In this case  $s_k$  has a very nice formula, which was discovered by Euler [3]:

$$(4.5) \quad \zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-1} B_{2k} \pi^{2k}}{(2k)!},$$

where  $B_{2k}$  is the  $2k$ -th Bernoulli number defined by

$$(4.6) \quad \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Then, apply (3.8) with  $c = -\pi^2$  to get

$$(4.7) \quad \frac{2^{2n-1}B_{2n}}{(2n)!} = \det \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \frac{1}{3!} \\ \frac{1}{3!} & 1 & 0 & \dots & 0 & \frac{1}{5!} \\ \frac{1}{5!} & \frac{1}{3!} & 1 & \dots & 0 & \frac{1}{7!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{(2n-3)!} & \frac{1}{(2n-5)!} & \frac{1}{(2n-7)!} & \dots & 1 & \frac{(n-1)}{(2n-1)!} \\ \frac{1}{(2n-1)!} & \frac{1}{(2n-3)!} & \frac{1}{(2n-5)!} & \dots & \frac{1}{3!} & \frac{n}{(2n+1)!} \end{pmatrix},$$

which is known.

4.2. **Bessel function**  $J_\nu(z)$ . Take

$$(4.8) \quad \lambda_k = \frac{1}{j_{\nu,k}^2}, \quad k \in \mathbb{N},$$

in Theorem 3.1, then,

$$(4.9) \quad f(z) = \frac{2^\nu \Gamma(\nu+1) J_\nu(z^{1/2})}{z^{\nu/2}} = \prod_{n=1}^{\infty} \left( 1 - \frac{z}{j_{\nu,n}^2} \right)$$

has the series expansion

$$(4.10) \quad f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\nu+1) z^n}{n! 2^{2n} \Gamma(\nu+n+1)}.$$

Hence

$$(4.11) \quad \sum_{1 \leq k_1 < k_2 < \dots < k_n} \frac{1}{j_{\nu,k_1}^2 \cdot j_{\nu,k_2}^2 \cdots j_{\nu,k_n}^2} = \frac{\Gamma(\nu+1)}{n! 2^{2n} \Gamma(\nu+n+1)},$$

or

$$(4.12) \quad \sum_{1 \leq k_1 < k_2 < \dots < k_n} \frac{1}{j_{\nu,k_1}^2 \cdot j_{\nu,k_2}^2 \cdots j_{\nu,k_n}^2} = \frac{1}{n! 4^n (\nu+1)_n}.$$

Then, from (3.8) with  $c = -\frac{1}{4}$  to obtain

$$(4.13)$$

$$\sum_{k=1}^{\infty} \frac{4^n (-1)^{n-1}}{j_{\nu,k}^{2n}} = \det \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \frac{1}{(\nu+1)} \\ \frac{1}{(\nu+1)} & 1 & 0 & \dots & 0 & \frac{1}{(\nu+1)_2} \\ \frac{1}{2!(\nu+1)_2} & \frac{1}{(\nu+1)} & 1 & \dots & 0 & \frac{1}{2!(\nu+1)_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{(n-2)!(\nu+1)_{n-2}} & \frac{1}{(n-3)!(\nu+1)_{n-3}} & \frac{1}{(n-4)!(\nu+1)_{n-4}} & \dots & 1 & \frac{1}{(n-2)!(\nu+1)_{n-1}} \\ \frac{1}{(n-1)!(\nu+1)_{n-1}} & \frac{1}{(n-2)!(\nu+1)_{n-2}} & \frac{1}{(n-3)!(\nu+1)_{n-3}} & \dots & \frac{1}{(\nu+1)} & \frac{1}{(n-1)!(\nu+1)_n} \end{pmatrix}.$$

Here are the first few  $s_n$ s, [15],

$$(4.14) \quad s_1 = \frac{1}{4(\nu+1)_1},$$

$$(4.15) \quad s_2 = \frac{1}{4^2 \prod_{j=1}^2 (\nu+1)_j},$$

$$(4.16) \quad s_3 = \frac{2(\nu+2)_1}{4^3 \prod_{j=1}^3 (\nu+1)_j},$$

$$(4.17) \quad s_4 = \frac{(5\nu+11)(\nu+2)_2}{4^4 \prod_{j=1}^4 (\nu+1)_j},$$

$$(4.18) \quad s_5 = \frac{2(7\nu+19)(\nu+2)_2(\nu+2)_3}{4^5 \prod_{j=1}^5 (\nu+1)_j}.$$

4.3. **Airy function**  $A(z)$ . Take

$$(4.19) \quad \lambda_k = \frac{1}{i_k^2}, \quad k \in \mathbb{N}.$$

From (2.21) and (2.22) to obtain

$$(4.20) \quad \sum_{1 \leq k_1 < k_2 < \dots < k_n} \frac{1}{i_{k_1}^2 \cdot i_{k_2}^2 \cdots i_{k_n}^2} = \frac{\sqrt{3}\Gamma\left(\frac{2}{3}\right)^2}{\sqrt[3]{4}\pi} \frac{16^{n/3}\Gamma\left(\frac{n}{3} + \frac{1}{6}\right)\Gamma\left(\frac{n}{3} + \frac{1}{2}\right)}{(2n)!}.$$

Let

$$(4.21) \quad a(n) = \frac{\sqrt{3}\Gamma\left(\frac{2}{3}\right)^2}{\sqrt[3]{4}\pi} \frac{\Gamma\left(\frac{n}{3} + \frac{1}{6}\right)\Gamma\left(\frac{n}{3} + \frac{1}{2}\right)}{(2n)!},$$

then,

$$(4.22) \quad \sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{i_k^{2n} 2^{4n/3}} = \det \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & a(1) \\ a(1) & 1 & 0 & \dots & 0 & 2a(2) \\ a(2) & a(1) & 1 & \dots & 0 & 3a(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a(n-2) & a(n-3) & a(n-4) & \dots & 1 & (n-1)a(n-1) \\ a(n-1) & a(n-2) & a(n-3) & \dots & -a(1) & na(n) \end{pmatrix}.$$

The first five  $s_n$  are:

$$(4.23) \quad s_1 = \frac{\sqrt{3}\Gamma\left(\frac{2}{3}\right)^2\Gamma\left(\frac{5}{6}\right)}{2^{1/3}\sqrt{\pi}}$$

$$(4.24) \quad s_2 = \frac{3\Gamma\left(\frac{2}{3}\right)^4\Gamma\left(\frac{5}{6}\right)^2}{2^{2/3}\pi} - \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{3}\right)}{2\sqrt{3}},$$

$$(4.25) \quad s_3 = \frac{\pi}{90} + \frac{3\sqrt{3}\Gamma\left(\frac{2}{3}\right)^6\Gamma\left(\frac{5}{6}\right)^3}{2\pi^{3/2}} - \frac{3\Gamma\left(\frac{2}{3}\right)^3\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{5}{3}\right)}{42^{1/3}\sqrt{\pi}},$$

$$(4.26) \quad s_4 = \frac{1}{54}\Gamma\left(\frac{2}{3}\right)^4 + \frac{(56\pi - 5)\Gamma\left(\frac{2}{3}\right)^2\Gamma\left(\frac{5}{6}\right)}{12602^{1/3}\sqrt{3\pi}} \\ + \frac{92^{2/3}\Gamma\left(\frac{2}{3}\right)^8\Gamma\left(\frac{5}{6}\right)^4 - 22^{1/3}\sqrt{3}\pi\Gamma\left(\frac{2}{3}\right)^5\Gamma\left(\frac{5}{6}\right)^2\Gamma\left(\frac{5}{3}\right)}{4\pi^2},$$

$$(4.27) \quad s_5 = \frac{5\Gamma\left(\frac{2}{3}\right)^6\Gamma\left(\frac{5}{6}\right)}{362^{1/3}\sqrt{3\pi}} + \frac{(56\pi - 5)\Gamma\left(\frac{2}{3}\right)^4\Gamma\left(\frac{5}{6}\right)^2}{10082^{2/3}\pi} \\ - \frac{5\Gamma\left(\frac{2}{3}\right)^8\Gamma\left(\frac{5}{6}\right)^3}{4\pi^{3/2}} + \frac{9\sqrt{3}\Gamma\left(\frac{2}{3}\right)^{10}\Gamma\left(\frac{5}{6}\right)^5}{22^{2/3}\pi^{5/2}} + \frac{(1 - 36\pi)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{8}{3}\right)}{12960\sqrt{3}}.$$

4.4.  **$q$ -Bessel function  $J_\nu^{(2)}(z; q)$ .** Take

$$(4.28) \quad \lambda_k = \frac{1}{j_{\nu,k}^2(q)}, \quad k \in \mathbb{N}.$$

From

$$(4.29) \quad \frac{2^\nu(q; q)_\infty J_\nu^{(2)}(z^{1/2}; q)}{(q^{\nu+1}; q)_\infty z^{\nu/2}} = \prod_{n=1}^{\infty} \left(1 - \frac{z}{j_{\nu,n}^2}\right) \\ = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+\nu)} z^n}{(q, q^{\nu+1}; q)_n 2^{2n}},$$

to obtain

$$(4.30) \quad \sum_{1 \leq k_1 < k_2 < \dots < k_n} \frac{1}{j_{\nu,k_1}^2(q) \cdot j_{\nu,k_2}^2(q) \cdots j_{\nu,k_n}^2(q)} = \frac{q^{n(n+\nu)}}{4^n (q, q^{\nu+1}; q)_n}.$$

and

$$(4.31)$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{n-1} 4^n}{q^{n\nu} j_{\nu,k}^{2n}(q)} \\ = \det \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & b(1; q) \\ b(1; q) & 1 & 0 & \dots & 0 & 2b(2; q) \\ b(2; q) & b(1; q) & 1 & \dots & 0 & 3b(3; q) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b(n-2; q) & b(n-3; q) & b(n-4; q) & \dots & 1 & (n-1)b(n-1; q) \\ b(n-1; q) & b(n-2; q) & b(n-3; q) & \dots & b(1; q) & nb(n; q) \end{pmatrix},$$

where

$$(4.32) \quad b(n; q) := \frac{q^{n^2}}{(q, q^{\nu+1}; q)_n}.$$

The first three  $s_n$ s are:

$$(4.33) \quad s_1 = \frac{q^{\nu+1}}{4(1-q)(1-q^{\nu+1})},$$

$$(4.34) \quad s_2 = \frac{q^{2(\nu+1)}(1+2q-q^{\nu+2})}{4^2(1-q^2)(1-q^{\nu+1})(q^{\nu+1}; q)_2},$$

$$(4.35) \quad s_3 = \frac{q^{3(\nu+1)}(1+3q+3q^2+3q^3-q^{\nu+2}-q^{\nu+3}-3q^{\nu+4}+q^{2\nu+5})}{4^3(1-q^3)(1-q^{\nu+1})^2(q^{\nu+1}; q)_3}.$$

**4.5. Ramanujan's entire function  $A_q(z)$ .** Take

$$(4.36) \quad \lambda_k = \frac{1}{i_k(q)}, \quad k \in \mathbb{N}.$$

From

$$(4.37) \quad A_q(z) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-1)^n z^n}{(q; q)_n} = \prod_{n=1}^{\infty} \left(1 - \frac{z}{i_n(q)}\right)$$

to get

$$(4.38) \quad \sum_{1 \leq k_1 < k_2 < \dots < k_n} \frac{1}{i_{k_1}(q) \cdot i_{k_2}(q) \cdots i_{k_n}(q)} = \frac{q^{n^2}}{(q; q)_n},$$

and

$$(4.39) \quad \sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{i_k^n(q)} = \det \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \frac{q}{1-q} \\ \frac{q}{1-q} & 1 & 0 & \dots & 0 & \frac{2q^4}{(q; q)_2} \\ \frac{q}{1-q} & \frac{q}{1-q} & 1 & \dots & 0 & \frac{3q^9}{(q; q)_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{q^{(n-2)^2}}{(q; q)_{n-2}} & \frac{q^{(n-3)^2}}{(q; q)_{n-3}} & \frac{q^{(n-4)^2}}{(q; q)_{n-4}} & \dots & 1 & \frac{(n-1)q^{(n-1)^2}}{(q; q)_{n-1}} \\ \frac{q^{(n-1)^2}}{(q; q)_{n-1}} & \frac{q^{(n-2)^2}}{(q; q)_{n-2}} & \frac{q^{(n-3)^2}}{(q; q)_{n-3}} & \dots & \frac{q}{1-q} & \frac{nq^{n^2}}{(q; q)_n} \end{pmatrix}.$$

The first five  $s_n$ s are:

(4.40)

$$s_1 = \frac{q}{1-q},$$

(4.41)

$$s_2 = \frac{q^2(1+2q)}{1-q^2},$$

(4.42)

$$s_3 = \frac{q^3(1+3q+3q^2+3q^3)}{1-q^3},$$

(4.43)

$$s_4 = \frac{q^4(1+2q+2q^3)(1+2q+2q^2+2q^3)}{1-q^4},$$

(4.44)

$$s_5 = \frac{q^5}{1-q^5} \times (1+5q+10q^2+15q^3+20q^4+20q^5+20q^6+15q^7+10q^8+5q^9+5q^{10}).$$

4.6. **Riemann zeta function**  $\zeta(s)$ . Take

$$(4.45) \quad \lambda_k = \frac{1}{z_k^2}, \quad k \in \mathbb{N}.$$

From (2.46) to obtain

$$(4.46) \quad \sum_{1 \leq k_1 < k_2 < \dots < k_n} \frac{1}{z_{k_1}^2 \cdot z_{k_2}^2 \cdot \dots \cdot z_{k_n}^2} = \frac{b_n}{(2n)!b_0},$$

and

$$(4.47) \quad \sum_{k=1}^{\infty} \frac{(-1)^{n-1} b_0^n}{z_k^{2n}} = \det \begin{pmatrix} b_0 & 0 & 0 & \dots & 0 & \frac{b_1}{2!} \\ \frac{b_1}{2!} & 1 & 0 & \dots & 0 & \frac{2b_2}{4!} \\ \frac{b_2}{4!} & \frac{b_1}{2!} & 1 & \dots & 0 & \frac{3b_3}{6!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{b_{n-2}}{(2n-4)!} & \frac{b_{n-3}}{(2n-6)!} & \frac{b_{n-4}}{(2n-8)!} & \dots & 1 & \frac{(n-1)b_{n-1}}{(2n-2)!} \\ \frac{b_{n-1}}{(2n-2)!} & \frac{b_{n-2}}{(2n-4)!} & \frac{b_{n-3}}{(2n-6)!} & \dots & \frac{b_1}{2!} & \frac{nb_n}{(2n)!} \end{pmatrix}.$$

The first four  $s_n$ s are:

$$(4.48) \quad s_1 = \frac{b_1}{2b_0},$$

$$(4.49) \quad s_2 = \frac{3b_1^2 - b_0b_2}{12b_0^2},$$

$$(4.50) \quad s_3 = \frac{30b_1^3 - 15b_0b_1b_2 + b_0^2b_3}{240b_0^3},$$

$$(4.51) \quad s_4 = \frac{630b_1^4 - 420b_0b_1^2b_2 + 35b_0^2b_2^2 + 28b_0^2b_1b_3 - b_0^3b_4}{10080b_0^4}.$$

The Dirichlet  $L$  series have similar formulas with  $z_k$  being replaced by  $z_k(\chi, a)$ , and  $b_k$  by  $b_k(\chi, a)$ .

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#### REFERENCES

- [1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Tenth Printing with corrections, New York, 1972.
- [2] G. E. Andrews, Ramanujan's "Lost" notebook IX: The entire Rogers–Ramanujan function, *Adv. Math.* 191 (2005), pp. 408–422.
- [3] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
- [4] Philippe Biane, Jim Pitman and Marc Yor, Probability Laws Related to the Jacobi Theta and Riemann Zeta Functions, and Brownian Excursions, *Bulletin (New Series) of the American Mathematical Society*, Volume 38, Number 4, Pages 435–465.
- [5] J. M. Borwein, D. M. Bradley, D. J. Broadhurst, and P. Lisonek, Combinatorial Aspects of Multiple Zeta Values, *Electronic J. Combinatorics* 5 (1998), R38.
- [6] H. Davenport, *Multiplicative Number Theory*, Springer-Verlag, New York,
- [7] A. Erdélyi, *Higher Transcendental Functions*, Vol.I, Vol.II, Vol.III, Robert E. Krieger Publishing Company, Malabar, Florida, 1985.
- [8] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.
- [9] W.K. Hayman, On the zeros of a q-Bessel function, *Complex Analysis and Dynamical Systems*, Contemp. Math., Amer. Math. Soc., Providence, RI (2006), pp. 205–216.
- [10] L. K. Hua, *Introduction to Number Theory*, Springer-Verlag, New York, 1982.
- [11] Mourad E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, Cambridge, 2005.
- [12] Olivier Vallée and Manuel Soares, *Airy Functions and Applications to Physics*, World Scientific, Singapore, 2004.
- [13] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society, Providence, Rhode Island 1939.
- [14] E. C. Titchmarsh, *The Theory of Riemann Zeta Function*, second edition, Clarendon Press, New York, 1987.
- [15] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1944.

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